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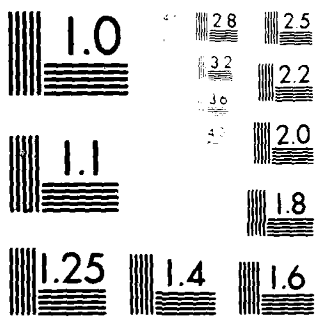

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The p-version of the finite element method is a new approach to finite element analysis which has been demonstrated to lead to significant computational savings, often by orders of magnitude (this approach was formerly called the constraint method; the new term p-version is more descriptive). Conventional approaches (called the h-version) generally employ low order polynomials as basis functions. Accuracy is achieved by suitably refining the approximating mesh. The p-version uses polynomials of arbitrary order $p \geq 2$ for problems in

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plane elasticity where C0 continuity is required and polynomials of order $p \geq 5$ for problems in plate bending where C1 continuity is required.

Hierarchic elements which implement the p-version efficiently are used together with precomputed arrays of elemental stiffness matrices.

Some important research results are:

1. In polygonal regions, under conditions usually satisfied in practice, if the h-version converges with order of error in energy norm $O(1/N^{\alpha})$, then the p-version converges with order of error in energy norm $O(1/N^{2\alpha})$, where N is the number of degrees of freedom. This applies to planar problems which require either C0 or C1 global continuity.
2. Hierarchic C0 elements for triangles or rectangles have been developed and implemented for $p \geq 2$. Hierarchic C1 triangular elements have been developed and implemented for $p \geq 5$. Hierarchic C0 solid elements have been developed for $p \geq 2$ for bricks, tetrahedra, triangular prisms, and rectangular pyramids.
3. Modified Bernstein polynomials have been constructed over triangles which provide a smooth approximation to functions in H_0^2. These polynomials are used to prove that the p-version of the finite element method converges in the C1 case (plate bending).
4. Explicit mappings have been constructed which map triangles with one curved side into the standard triangle. These mappings have the property that the resulting elemental stiffness matrices can be integrated in closed form. The need for numerical quadrature, which may be inefficient in the p-version, is thereby obviated. The curved side can be either parabolic or elliptic. The newly constructed mappings permit the p-version to be applied to domains with curved boundaries of specified shapes.

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SUMMARY

The conventional approach to finite element analysis is called the h-version of the finite element method. In this approach the domain is subdivided into smaller elements (usually triangles or rectangles in the plane, and bricks or tetrahedra in three dimensions). Accuracy is obtained by fixing the degree p of the local polynomial basis function (p is usually taken to be 2 or 3) and allowing h, the maximum diameter of the elements, to go to zero. This is in constrast to the constraint method for finite elements, now called the p-version of the finite element method, in which the subdivision is kept fixed but p is allowed to go to infinity. In order to implement the p-version efficiently new families of finite elements are required with the property that as much computation as possible can be retained from the p-th degree approximation when passing to the (p+1)st degree approximation. Such families of finite elements are called hierarchic. When using such families the elemental stiffness matrix corresponding to a pth degree approximation is a submatrix of the elemental stiffness matrix corresponding to a (p+1)st degree approximation.

The p-version uses polynomials of degree $p \geq 2$ for problems which require C^0 continuity (e.g. planar elasticity) and polynomials of degree $p \geq 5$ for problems which require C^1 continuity (e.g. plate bending).

Some significant research results which were obtained are the following:

1. Rate of convergence of the p-version in the C^0 case (singularity problem). Let Ω be a bounded polygonal domain in the plane and consider the model problem

$$\begin{aligned} -\Delta u + u &= f \quad \text{on } \Omega \\ \Gamma u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where $\Gamma u = u$ or $\Gamma u = \partial u/\partial n$. Let α_i be the angle made by $\partial\Omega$ at a vertex A_i, $1 \leq i \leq n$. Let u_p be the solution of (1) using the p-version, let u_h be the solution of (1) using the h-version, and let u_0 be the true solution to (1) in the weak sense. Suppose N is the number of degrees of freedom in the two finite element solutions u_p, u_h. Then under conditions usually satisfied in practical applications, we have

$$||u_h - u_0||_{1,\Omega} = O(N^{-\frac{1}{2}\gamma}) \quad , \quad ||u_p - u_0||_{1,\Omega} = O(N^{-\gamma}) \tag{2}$$

where

$$\gamma = \min \gamma_i \qquad \gamma_i = \pi/\alpha_i \ , \ 1 \leq i \leq n$$

2. Rate of convergence of the p-version in the C^1 case (singularity problem).

Let Ω, u_p, u_h, α_i, $1 \leq i \leq n$, be as before and consider the model problem

$$\begin{aligned} &\Delta^2 u = f \quad \text{on } \Omega \\ &u = 0 \qquad \partial u/\partial n = 0 \quad \text{on } \partial\Omega \quad \text{(clamped edge)} \end{aligned} \tag{3}$$

Let γ_i be the smallest root of the equation

$$\sin^2(\gamma-1)\alpha_i - (\gamma-1)^2\sin^2\alpha_i = 0$$

for $1 \leq i \leq n$ and let $\gamma = \min \gamma_i$. Then, the conclusion in (2) holds.

These two results taken together show that for two dimensional problems the p-version has twice the rate of convergence of the h-version. This result has been demonstrated computationally in extensive numerical experimentation.

3. Hierarchic C^0 families of triangles and rectangles have been developed and implemented for planar problems for $p \geq 2$. Hierarchic C^1 families

of triangles, which use corrective rational functions to enforce conformity in a minimal sense, have been developed and implemented for planar problems for $p \geq 5$. Hierarchic families of solid elements for $p \geq 5$ have been developed for bricks, tetrahedra, triangular prisms, and rectangular pyramids.

4. Mappings of triangles with one curved boundary into the standard triangle have been constructed with the property that the resulting elemental stiffness matrices can be integrated in closed form. This is important because in the p-version elements are taken to be large so that numerical quadrature may be inefficient. The mappings which have been constructed are for two cases: curved side parabolic, straight sides arbitrary; and curved sides elliptic, straight sides parallel to the axis of the ellipse. These mappings make it possible for the p-version to be applied when the domain Ω is not necessarily polygonal.

5. *Modified* Bernstein polynomials have been developed. These polynomials are defined over triangles in terms of natural coordinates, and they enforce C^1 continuity across boundaries of triangles. The modified Bernstein polynomials can be used to prove the convergence of the p-version of the finite element method in the C^1 case.

PROFESSIONAL PERSONNEL

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PAPERS PUBLISHED AND PRESENTED SINCE THE START OF THE PROJECT (1976)

Published Papers:

1. "Hierarchal Finite Elements and Precomputed Arrays", by Mark P. Rossow and I. Norman Katz, Int. J for Num. Method in Engr., Vol. 13, No. 6 (1978) pp. 977-999.

2. "Nodal Variables for Complete Conforming Finite Elements of Arbitrary Polynomial Order", by I. Norman Katz, A. G. Peano, and Mark P. Rossow, Computers and Mathematics with Applications, Vol. 4, No. 2, (1978), pp. 85-112.

3. "P-convergent Finite Element Approximations in Linear Elastic Fracture Mechanics", by Anil K. Mehta (doctoral dissertation), Department of Civil Engineering, Washington University (1978).

4. "An Improved P-version Finite Element Algorithm and a Convergence Result for the P-version" by Anthony G. Kassos, Jr. (doctoral dissertation) Department of Systems Science and Mathematics, Washington University, (August, 1979).

5. "Hierarchic Families for the P-version of the Finite Element Method", I. Babuska, I. N. Katz and B. A. Szabo, invited paper presented at the Third IMACS International Symposium on Computer Methods for Partial Differential Equations, published in Advances in Computer Methods for Partial Differential Equations - III (1979), pp. 278-286.

6. "The P-version of the Finite Element Method", I. Babuska, B. A. Szabo, and I. N. Katz, SIAM J. Numer. Anal. Vol. 18, No. 3, June 1981, pp. 515-545.

7. "Implementation of a C^1 Triangular Element Based on the P-version of the Finite Element Method", by Douglas W. Wang, I. Norman Katz, and Barna A. Szabo, to be presented at the Symposium on Advances and Trends in Structural and Solid Mechanics, Washington D.C., October 4-7, 1982 and to appear in the proceedings of the Symposium.

8. "Hierarchic Families of Complete Conforming Solid Finite Elements of Various Shapes", by I. Norman Katz, (in preparation).

9. "The P-version of the Finite Element Method for Problems Requiring Smooth Solutions" by Douglas W. Wang (doctoral dissertation), Department of Systems Science and Mathematics, Washington University (in preparation).

Presented Papers:

10. "Hierarchical Approximation in Finite Element Analysis", by I. Norman Katz, International Symposium on Innovative Numerical Analysis in Applied Engineering Science, Versailles, France, May 23-27, 1977.

11. "Efficient Generation of Hierarchal Finite Elements Through the Use of Precomputed Arrays", by M. P. Rossow and I. N. Katz, Second Annual ASCE Engineering Mechanics Division Speciality Conference, North Carolina State University, Raleigh, North Carolina, May 23-25, 1977.

12. "C^1 Triangular Elements of Arbitrary Polynomial Order Containing Corrective Rational Functions", by I. Norman Katz, SIAM 1977 National Meeting, Philadelphia, Pennsylvania, June 13-15, 1977.

13. "Hierarchical Complete Conforming Tetrahedral Elements of Arbitrary Polynomial Order", by I. Norman Katz, presented at SIAM 1977 Fall Meeting, Albuquerque, New Mexico, October 31-November 2, 1977.

14. "A Hierarchical Family of Complete Conforming Prismatic Finite Elements of Arbitrary Polynomial Order", by I. Norman Katz, presented at SIAM 1978 National Meeting, Madison, Wisconsin, May 24-26, 1978.

15. "Comparative Rates of h- and p-Convergence in the Finite Element Analysis of a Model Bar Problems", by I. Norman Katz, presented at the SIAM 1978 Fall Meeting, Knoxville, Tennessee, October 20-November 1, 1978.

16. "Smooth Approximation to a Function in $H_0^2(D)$ by Modified Bernstein Polynomials Over Triangles" by A. G. Kassos, Jr., and I. N. Katz, presented at the SIAM 1979 Fall Meeting, Denver, Colorado, November 12-14, 1979.

17. "Triangles With One Curved Side for the P-version of the Finite Element Method" by I. Norman Katz, presented at the SIAM 1980 Spring Meeting, Alexandra, Virginia, June 5-7, 1980.

18. "The Rate of Convergence of the P-version of the Finite Element Method for Plate Bending Problems", by Douglas W. Wang and I. Norman Katz, presented at the SIAM 1981 Fall Meeting, Cincinnati, Ohio, October 26-28, 1981.

SEMINARS PRESENTED AT GOVERNMENT LABORATORIES

1. "Advanced Stress Analysis Technology" by B. A. Szabo and I. N. Katz, presented on September 8, 1977 at the Air Force Flight Dynamic Laboratory, Wright-Patterson Air Force Base.

Abstract

With one exception, all finite element software systems have element libraries in which the approximation properties of elements are frozen. The user controls only the number and distribution of finite elements. The exception is an experimental software system, developed at Washington University. This system, called COMET-X, employs conforming elements based on complete polynomials of arbitrary order. The elements are hierarchic, i.e. the stiffness matrix of each element is embedded in the stiffness matrices of all higher order elements of the same kind. The user controls not only the number and distribution of finite elements but their approximation properties as well. Thus convergence can be achieved on fixed mesh. This provides for very efficient and highly accurate approximation and a new method for computing stress intensity factors in linear elastic fracture mechanics. The theoretical developments are outlined, numerical examples are given and the concept of an advanced self-adaptive finite element softward system is presented.

2. "The Constraint Method for Finite Element Stress Analysis", by I. N. Katz, presented at the National Bureau of Standards, Applied Mathematics Division on October 19, 1977.

Abstract

In conventional approaches to finite element stress analysis accuracy is obtained by fixing the degree p of the approximating polynomial and by allowing the maximum diameter h of elements in the triangulation to approach zero. An alternate approach is to fix the triangulation and to increase the degrees of approximating polynomails in those elements where more accuracy is required. In order to implement the second approach efficiently it is necessary to have a family of finite elements of arbitrary polynomial degree p with the property that as much information as possible can be retained from the pth degree approximation when computing the (p+1)st degree approximation. Such a HIERARCHIC family has been formulated with $p \geq 2$ for problems in plane stress analysis and with $p \geq 5$ for problems in plate bending. The family is described and numerical examples are presented which illustrates the efficiency of the new method.

3. "The P-version of the Finite Element Method", by I. N. Katz and B. A. Szabo, presented at Air-Force Flight Dynamics Laboratory Wright-Patterson Air Force Base on April 23, 1981.

Abstract

The theoretical basis of the p-version of the finite element method has been established only quite recently. Nevertheless, the p-version is already seen to be the most promising approach for implementing adaptivity in practical computations. The main theorems establishing asymptotic rates of convergence for the p-version, some aspects of the algorithmic structure of p-version computer codes, numerical experience and a posteriori error estimation will be discussed from the mathematical and engineering points of view.

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